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WEYL'S THEOREM FOR ALGEBRAICALLY TOTALLY $p-(\alpha, \beta)$-NORMAL OPERATORS

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Abstract

An operator $T \in B(\mathcal{H})$ is said to be $p-(\alpha, \beta)$ normal operators for $0 < p \leq 1$ if $\alpha^2(T^*T)^p \leq (TT^*)^p \leq \beta^2(T^*T)^p$, $0 \leq \alpha \leq 1 \leq \beta$. In this paper, we prove Weyl's theorem for totally $p-(\alpha, \beta)$-normal operators and algebraically totally $p-(\alpha, \beta)$-normal operators.

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1. Introduction and Preliminaries

Let $B(\mathcal{H})$ denotes the algebra of all bounded linear operators acting on an infinite dimensional separable Hilbert space $\mathcal{H}$. For positive operators A and B, write $A \geq B$ if $A - B \geq 0$. If A and B are invertible and positive operators, it is well known that $A \geq B$ implies that $\log A \geq \log B$. However [3], $\log A \geq \log B$ does not necessarily imply $A \geq B$. A result due to Ando [4] states that for invertible positive operators A and B, $\log A \geq \log B$, if and only if $A^r \geq (A^{\frac{r}{2}} B^r A^{\frac{r}{2}})^{\frac{1}{2}}$ for all $r \geq 0$. For an operator T, let $U|T|$ denote the polar decomposition of T, where U is a partially isometric operator, $|T|$ is a positive square root of T^*T and $\ker(T) = \ker(U) = \ker(|T|)$, where $\ker(S)$ denotes the kernel of operator S.

An operator $T \in B(\mathcal{H})$ is positive, $T \geq 0$, if $(Tx, x) \geq 0$ for all $x \in \mathcal{H}$, and posinormal if there exists a positive $\lambda \in B(\mathcal{H})$ such that $TT^* = T^* \lambda T$. Here λ is called interrupter of T. In the other words, an operator T is called posinormal if $TT^* \leq c^2 T^* T$, where T^* is the adjoint of T and $c > 0$ [11]. An operator T is said to be heminormal if T is hyponormal and T^*T commutes with TT^*. An operator T is said to be p-posinormal if $(TT^*)^p \leq c^2 (T^*T)^p$ for some $c > 0$. It is clear that 1-posinormal is posinormal. An operator T is said to be p-hyponormal, for $p \in (0, 1)$, if $(T^*T)^p \geq (TT^*)^p$. A 1-hyponormal operator is hyponormal which has been studied by many authors and it is known that hyponormal operators have many interesting properties similar to those of normal operators [34]. Fututa et al. [17], have characterized class A operator as follows. An operator T belongs to class A if and only if $(T^*|T|T)^{\frac{1}{2}} \geq T^*T$.

An operator T is called normal if $T^*T = TT^*$ and (p, k)-quasihyponormal if $T^{*^k}((T^*T)^p - (TT^*)^p)T^k \geq 0\,(0 < p \leq 1, k \in \mathbb{N})$. Aluthge [2], Gupta [9], Arora and Arora [5] introduced p-hyponormal, p-quasihyponormal and k-quasihyponormal operators, respectively.

$$p\text{-hyponormal} \subset p\text{-posinormal} \subset (p, k)\text{-quasiposinormal},$$

$$p\text{-hyponormal} \subset p\text{-quasihyponormal}$$

$$\subset (p, k)\text{-quasihyponormal} \subset (p, k)\text{-quasiposinormal},$$

and

$$\text{hyponormal} \subset k\text{-quasihyponormal} \subset (p, k)\text{-quasihyponormal}$$

$$\subset (p, k)\text{-quasiposinormal},$$

for a positive integer k and a positive number $0 < p \leq 1$.

An operator $T \in B(\mathcal{H})$ is said to be (α, β)-normal operators, if $\alpha^2 T^*T \leq TT^* \leq \beta^2 T^*T$, $0 \leq \alpha \leq 1 \leq \beta$ [25]. The example of an M-hyponormal operator given by Wadhwa [32], the weighted shift operator defined by $Te_1 = e_2$, $Te_2 = 2e_3$, and $Te_i = e_{i+1}$ for $i \geq 0$, is not a $p - (\alpha, \beta)$-normal, which is neither normal nor hyponormal. So it is clear that the class of $p - (\alpha, \beta)$-normal lies between hyponormal and M-hyponormal operators. Now, the inclusion relation becomes

$$\text{normal} \subseteq \text{hyponormal} \subseteq (\alpha, \beta)\text{-normal}$$

$$\subseteq p - (\alpha, \beta)\text{-normal} \subseteq M\text{-hyponormal} \subseteq \text{dominant}.$$

Let $B(\mathcal{H})$ and $K(\mathcal{H})$ denote, respectively, the algebra of bounded linear operators and the ideal of compact operators acting on an infinite dimensional separable Hilbert space $\mathcal{H}$. If $T \in B(\mathcal{H})$, we shall write $\mathrm{N(A)}$ and $\mathrm{R(T)}$ for the null space and the range of T, respectively. Also, let $\alpha(T) := \dim N(T)$, $\beta(T) := \dim N(T^*)$, and let $\sigma(T)$, $\sigma_a(T)$, and $\prod_0(T)$ denote the spectrum, approximate point spectrum, and the point spectrum of T, respectively. An operator $T \in B(\mathcal{H})$ is called Fredholm if it has a closed range, a finite dimensional null space, and its range has finite co-dimension. The index of a Fredholm operator is given by

$$I(A) := \alpha(T) - \beta(T).$$

For $T \in B(\mathcal{H})$, let $\alpha(T) = \dim(\ker(T))$, $\beta(T) = \dim(\ker(T^*))$, and $\sigma(T)$, $\sigma_a(T)$, $\pi_0(T)$ denote the *spectrum*, *approximate point spectrum* and the *point spectrum* of T, respectively. An operator $T \in B(\mathcal{H})$ is called *Fredholm* if it has closed range, finite dimensional null space and its range has finite co-dimension. The *index* of a Fredholm operator is given by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$.

A bounded linear operator T is called *Weyl* if it is Fredholm of index zero and *Browder* if it is Fredholm of finite ascent and descent; equivalently, if T is Fredholm and $T - \lambda$ is invertible for sufficiently small $|\lambda| \geq 0$, $\lambda \in \mathbb{C}$ [19]. For $T \in B(\mathcal{H})$, for each non-negative integer n, define T_n to be the restriction of T to $\operatorname{ran}(T^n)$ into $\operatorname{ran}(T^n)$. If for some n, the space $\operatorname{ran}(T^n)$ is closed and T_n is a Fredholm operator, then T is called a *B-Fredholm* operator [6]. $T \in B(\mathcal{H})$ is called a *B-Weyl* operator if it is a B-Fredholm operator of index zero.

$$\text{Let } \Phi_+(\mathcal{H}) = \{T \in B(\mathcal{H}) : \alpha(T) < \infty \text{ and } T(\mathcal{H}) \text{ is closed}\}.$$

$$\text{Let } \Phi_-(\mathcal{H}) = \{T \in B(\mathcal{H}) : \beta(T) < \infty\},$$

denote the class of all *upper semi-Fredholm* operators and *lower semi-Fredholm* operators. And let $\Phi_+^-(\mathcal{H})$ is the class of all *left-semi-Fredholm* operators, such that for every $T \in \Phi_+^-(\mathcal{H})$, ind $T \leq 0$.

Let SBF_+ be the class of all *upper semi-B-Fredholm* operators, SBF_+^- be the class of all *semi-B-Fredholm* operators such that for every $T \in SBF_+^-$, $\text{ind}(T) \leq 0$. The *essential spectrum* $\sigma_e(T)$, the *Weyl spectrum* $\sigma_w(T)$ and the *Browder spectrum* $\sigma_b(T)$ of T are defined by [18], [19]:

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\};$$

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\};$$

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\};$$

respectively. And the *B-Weyl spectrum*, *a-Browder spectrum*, *essential approximate spectrum*, and *Weyl* (*essential*) *approximate point spectrum* are defined by

$$\sigma_{Bw}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{-Weyl}\};$$

$$\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_+^-(\mathcal{H})\};$$

$$\sigma_{ab}(T) = \bigcap \sigma_a(T + K) : TK = KT,\ K \text{ is a compact operator};$$

$$\sigma_{wa}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_+(\mathcal{H}) \text{ and } \text{ind}(T - \lambda) \leq \infty\}.$$

Evidently, $\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup acc\, \sigma(T)$;

where we write $acc\, \sigma(T)$ for the accumulation points of $\sigma(T)$.

If we write $isoK = K \setminus accK$, then we write

$$\pi_{00}(T) = \{\lambda \in iso\, \sigma(T) : 0 < \alpha(T - \lambda) < \infty\},$$

$$p_{00}(T) = \sigma(T) \setminus \sigma_b(T),$$

$$\pi_{00}^a(T) = \{\lambda \in iso\, \sigma_a(T) : 0 < \alpha(T - \lambda) < \infty\}.$$

We say that *Weyl's theorem* holds for $T \in B(H)$, if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T).$$

We say that *Generalized Weyl's theorem* holds for $T \in B(H)$, if

$$\sigma(T) \setminus \sigma_{BW}(T) = E^a(T).$$

We say that *a-Weyl's theorem* holds for $T \in B(H)$, if

$$\sigma_{ap}(T) \setminus \sigma_{wa}(T) = \pi_{00}(T).$$

We say that *Browder's theorem* holds for $T \in B(H)$, if

$$\sigma(T) \setminus \sigma_w(T) = p_{00}(T).$$

We say that *a-Browder's theorem* holds for $T \in B(H)$, if

$$\sigma_{ea}(T) = \sigma_{ab}(T).$$

We say that *generalized a-Weyl's theorem* holds for $T \in B(H)$, if

$$\sigma_{SBF_+^-}(T) = \sigma_{ap}(T) \setminus E^a(T),$$

where E^a is the set of all eigenvalues of T that are isolated in $\sigma_{ap}(T)$.

It is clear that [8]

generalized a-Weyl's theorem $\Rightarrow$ generalized Weyl's theorem

$\Rightarrow$ Weyl's theorem $\Rightarrow$ Browder's theorem;

and

generalized a-Weyl's theorem $\Rightarrow$ a-Weyl's theorem

$\Rightarrow$ Weyl's theorem $\Rightarrow$ Browder's theorem;

and

generalized a-Weyl's theorem $\Rightarrow$ a-Weyl's theorem

$\Rightarrow$ generalized Browder's theorem $\Rightarrow$ Browder's theorem.

Mecheri [24] has proved generalized a-Weyl's theorem for some classes of operators. Here in this paper, we prove generalized a-Weyl's theorem for totally $p-(\alpha, \beta)$-normal operators.

We say that $T \in B(H)$ has the single valued extension property (SVEP) if, for every open set $U \subseteq \mathbb{C}$, the only analytic function $f : U \to H$ that satisfies the equation $(T - \lambda)f(\lambda) = 0$ is the constant function $f \equiv 0$.

An operator T is said to be class $\mathcal{Y}_\alpha$ for $\alpha \geq 1$, if there exists a positive number k_α such that

$$|TT^* - T^*T|^\alpha \leq k_\alpha^2 (T - \lambda)^* (T - \lambda) \text{ for all } \lambda \in \mathbb{C}.$$

It is known that $\mathcal{Y}_\alpha \subset \mathcal{Y}_\beta$ if $1 \leq \alpha \leq \beta$. Let $\mathcal{Y} = \bigcup_{1 \leq \alpha} \mathcal{Y}_\alpha$. We remark that a class $\mathcal{Y}_\infty$ operator T is M-hyponormal, i.e., there exists a positive number M such that

$$(T - \lambda)(T - \lambda)^* \leq M^2 (T - \lambda)^* (T - \lambda) \text{ for all } \lambda \in \mathbb{C},$$

and M-hyponormal operators are class $\mathcal{Y}_\in$ [31].

An operator T is said to have a finite ascent if $\ker T^m = \ker T^{m+1}$ for some positive integer m, and finite descent if $\operatorname{ran} T^n = \operatorname{ran} T^{n+1}$ for some positive integer n.

Let X be a Banach space. An operator $T \in B(X)$ is called B-Fredholm by Berkani [6], if there exists $n \in \mathcal{N}$ for which the induced operator

$$T_n : T^n(X) \to T^n(X),$$

is Fredholm in the usual sense, and B-Weyl if in addition T_n has index zero.

Weyl [33] has proved Weyl's theorem for Hermitian operators. From then on, it has been extended to various classes of operators such as hyponormal, Toeplitz [10], and to several classes operators including hyponormal operators [26, 30]. Curto and Han [12] have proved the theorem for algebraically paranormal operators. Generalization of Weyl's theorem named as generalized Weyl's theorem is proved for hyponormal operators by Berkani [7]. Recently, it has been extended to various classes of operators such as algebraically hyponormal [18], algebraically class A [27], etc..

Here in this paper, we extend the same for more general class of operators, namely, totally $p-(\alpha, \beta)$-normal operators. Also, we prove Weyl's theorem for algebraically totally $p-(\alpha, \beta)$-normal operators.

Dragomir and Moslehian [13] has given various inequalities between the operator norm and numerical radius of (α, β)-normal operators. In this article, we extend the results to $p-(\alpha, \beta)$-normal operators.

2. Generalized Weyl Theorem for Totally $p-(\alpha, \beta)$-Normal Operators

We prove the following lemmas to prove the generalized a-Weyl's theorem for totally $p-(\alpha, \beta)$-normal operators.

Lemma 2.1. *Let $T \in B(H)$ be totally $p-(\alpha, \beta)$-normal, then $T-\lambda$ has finite ascent for all then T has SVEP.*

Proof. If T is totally $p-(\alpha, \beta)$-normal, then $\ker(T-\lambda)^p = \ker(T-\lambda)^{*p}$, ascent $(T-\lambda) \leq p \leq 1$ for all $\lambda \in \mathbb{C}$, then T has SVEP. □

Theorem 2.2. *Let $T \in B(H)$ be totally (α, β)-normal and $\lambda \in \sigma(T)$ be an isolated point of $\sigma(T)$, then $H_0(T) = E_\lambda H$, where E_λ denote the Riesz idempotent for λ.*

Proof. Since T has SVEP, by [28] the theorem follows. □

Theorem 2.3. *Let $T \in B(H)$ be totally $p-(\alpha, \beta)$-normal. Let $M \subseteq H$ be an invariant subspace of T, then the restriction $T|_M$ is also totally $p-(\alpha, \beta)$-normal.*

Proof. Let P be the orthogonal projection on $\mathcal{M}$. Then for all $z \in \mathbb{C}$ and for all $x \in \mathcal{M}$,

$$\|(T - zI)|_{\mathcal{M}})^* x\| = \|P(T - Z)^*\| = \|(T - zI)^* x\| = \mathcal{M}_z \|(A|_{\mathcal{M}} - zIx)\|.$$

$\square$

Lemma 2.4. *Let $T \in B(H)$ be totally $p-(\alpha, \beta)$-normal. If $\sigma(T - \lambda I) = \{0\}$, then $T = \lambda$.*

Proof. If T is totally $p-(\alpha, \beta)$-normal, then $\mathrm{T} \in \mathcal{Y}_{\in} \subseteq \mathcal{Y}$ by using the same argument used in [31, Lemma 10], and it is known that if $T \in \mathcal{Y}$ and if $\sigma(T) = \{0\}$, then $A = 0$ [31, Lemma 14]. Therefore, if T is totally $p-(\alpha, \beta)$-normal, then $T - \lambda I$ is also totally $p-(\alpha, \beta)$-normal and $\sigma(T - \lambda I) = \{0\}$. Hence $T - \lambda I = 0$. $\square$

Theorem 2.5. *Let T be a totally $p-(\alpha, \beta)$-normal operator and λ_0 be an isolated point of $\sigma(T)$. If E is the Riesz idempotent for λ_0, then E is self adjoint and $E(H) = \ker(T - \lambda_0) = \ker(T^* - \overline{\lambda_0})$.*

Proof. The second equality follows from the definition of totally (α, β)-normal operators. Any $p-(\alpha, \beta)$-normal operator T can be represented as a block matrix $\begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}$, where $C : \overline{\mathrm{ran}(T)} \longrightarrow \overline{\mathrm{ran}(T)}$ has zero kernel.

$$\begin{aligned} E &= \frac{1}{2\pi i} \int_{\partial D} (\lambda - T)^{-1} d\lambda \\ &= \frac{1}{2\pi i} \int_{\partial D} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & (\lambda - C)^{-1} \end{pmatrix} d\lambda \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

on $E(H) \oplus E(H)^{\perp}$.

Hence E is self adjoint. □

It is known that if $T \in B(H)$ has SVEP, then $f(T)$ has SVEP for each f analytic in an open neighbourhood of $\sigma(T)$.

Lemma 2.6. *Let $T \in B(H)$ be totally $p-(\alpha, \beta)$-normal. Then generalized Weyl's theorem hold for T.*

Proof. Assume that $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $T - \lambda I$ is B-Weyl and not invertible. We claim that $\lambda \in \partial\sigma(T)$. Assume to the contrary that λ is an interior point of $\sigma(T)$. Then, there exists a neighbourhood U of λ such that $\dim(T - \mu) > 0$ for all $\mu \in U$. It follows from [16, Theorem 10] that T does not have SVEP. On the other hand, since T is totally $p-(\alpha, \beta)$-normal, it follows from Lemma 2.1 above that T has SVEP, which is a contradiction. Therefore $\lambda \in \partial\sigma(T)$. Conversely, assume that $\lambda \in E(T)$, then λ is isolated in $\sigma(T)$. From [20, Theorem 7.1], we have $X = M \oplus N$, where M and N are closed subspaces of $X, U = (T - \lambda I)|_{\mathcal{M}}$ is an invertible operator and $V = (T - \lambda I)|_{\mathcal{N}}$ is a quasinilpotent operator. Since T is totally $p-(\alpha, \beta)$-normal, V is also totally $p-(\alpha, \beta)$-normal, and from Lemma 2.4 is nilpotent. Therefore $T - \lambda I$ is Drazin invertible [29, Proposition 6] and [21, Corollary 2.2]. By [6, Lemma 4.1], $T - \lambda I$ is a B-Fredholm operator of index 0. □

Theorem 2.7. *Let $T \in B(H)$ be totally $p-(\alpha, \beta)$-normal. Then $f(T)$ obeys generalized Weyl's theorem for every function f analytic in the neighbourhood of $\sigma(T)$.*

Proof. Since the operator T satisfies the generalized Weyl's theorem and it is isoloid, it follows from [7, Lemma 2.9] that $f(T)$ obeys the generalized Weyl's theorem. □

Theorem 2.8. *Let $T \in B(H)$ be totally $p-(\alpha, \beta)$-normal. Then $f(T))$ satisfies Browder's theorem for each function f analytic in a neighbourhood of $\sigma(T)$.*

Proof. It is known that operators with SVEP satisfy Browder's theorem [14]. Then $f(T)$ satisfies Browder's theorem. $\square$

Theorem 2.9. *Let T be totally $p-(\alpha, \beta)$-normal. Then the generalized a-Weyl's theorem holds for T.*

Proof. We have to prove that $\sigma_{ap}(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T)$. For this, assume that $\lambda \in \sigma_{ap}(T) \setminus \sigma_{SBF_+^-}(T)$. Then $T - \lambda I$ is an upper semi-B-Fredholm operator and $\operatorname{ind}(T - \lambda I) \leq 0$. Hence, for n large enough, $T - (\lambda + \frac{1}{n})I$ is an upper semi-Fredholm operator and $\operatorname{ind}(T - (\lambda + \frac{1}{n})\ \ I) = indT - \lambda I$ [8]. Therefore $\operatorname{ind}(T - (\lambda + \frac{1}{n})I) \leq 0$. Since T has SVEP, [1] implies that $\operatorname{ind}(T - (\lambda + \frac{1}{n})I) \geq 0$. Thus $\operatorname{ind}(T - (\lambda + \frac{1}{n})I) = 0$. It follows that $\operatorname{ind}(T - \lambda I) = 0$. This implies that $T - \lambda I$ is a B-Fredholm operator of index zero. Since T has SVEP, we have $\sigma(T) = \sigma_{ap}(T)$ and we have, $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then it follows from Theorem 2.6 that $\lambda \in E(T)$. Hence $\lambda \in E^a(T)$. Conversely, let $\lambda \in E^a(T)$. Then λ is an isolated point of $\sigma_{ap}(T)$. Therefore $\overline{\lambda}$ is an isolated point of $\sigma(T^*)$. Let P be the spectral projection defined by

$$P = \int_{\partial B_0} (\lambda_0 I - T^*)^{-1} d\lambda_0,$$

where B_0 is an open disk centered at $\overline{\lambda}$ that contains no other points of $\sigma(T^*)$. Then T^* can be represented as the direct sum

$$T^* = T_1 \oplus T_2, \text{ where } \sigma(T_1) = \overline{\lambda} \text{ and } \sigma(T_2) = \sigma(T^*) \setminus \{\overline{\lambda}\}.$$

Then $\sigma(T^*) - T_2$ is invertible. We have to consider the two following cases:

Case 1. Assume that $\lambda = 0$. Then $\sigma(T_1) = \{0\}$. Since T_1 is a totally $p-(\alpha, \beta)$-normal operator, it follows that $T_1 = 0$ by Lemma 2.1. Therefore $\overline{\lambda}I - T^* = 0 \oplus \overline{\lambda}I - T_2$.

Case 2. Assume $\lambda \neq 0$. Since T_1 is an invertible totally $p-(\alpha, \beta)$-normal operator, it follows that T_1^{-1} is an invertible totally $p-(\alpha, \beta)$-normal operator. Then $\|T_1\| = |\lambda|$ and $\|T_1^{-1}\| = \frac{1}{\lambda}$. Therefore, for any $x \in R(P)$, we have

$$\|x\| \leq \|T_1^{-1}\| \|T_1 x\| = \frac{1}{|\lambda|}\|T_1 x\| \leq \frac{1}{|\lambda|}|\lambda| \|x\| = \|x\|.$$

Hence, $\frac{1}{\lambda}T_1$ is unitary. Therefore T_1 is normal and $\overline{\lambda}I - T_1$ is also normal. Since $\overline{\lambda}I - T_1$ is quasinilpotent and the only normal quasinilpotent operator is zero, it follows that $\overline{\lambda} - T^* = 0 \oplus \overline{\lambda}I - T_2$. Now since $\overline{\lambda}I - T_2$ is invertible, it is known that $\overline{\lambda}I - T^*$ has finite ascent and descent. Therefore $\lambda I - T$ has finite ascent and descent. This implies that $\lambda \in \sigma_{ap}(T) \setminus \sigma_{SBF_+^-}(T)$. □

Theorem 2.10. *Let T be a totally $p-(\alpha, \beta)$-normal operator. Then T is a-isoloid.*

Proof. Since T is a totally $p-(\alpha, \beta)$-normal operator, Theorem 2.9 implies that a-Weyl's theorem holds for T and $\sigma(T) = \sigma_{ap}(T)$. If we assume that $\lambda \in \operatorname{iso}\sigma_{ap}(T) = \operatorname{iso}\sigma(T)$, then $\overline{\lambda} \in \operatorname{iso}\sigma(T^*)$. Since T^* is a totally $p-(\alpha, \beta)$-normal operator and it is isoloid, $N(\overline{\lambda}I - T^*) \neq \{0\}$. Since $N(\overline{\lambda}I - T^*) \subseteq N(\lambda I - T)$, we have $N(\lambda I - T) \neq \{0\}$. Thus T is a-isoloid. □

Theorem 2.11. *Let T be totally $p-(\alpha, \beta)$-normal. Then $f(T)$ obeys the generalized a-Weyl's theorem for every function f analytic in a neighbourhood of $\sigma(T)$.*

Proof. Since T is a-isoloid, $T \in T_2(H)$ and T obeys the generalized a-Weyl's theorem, [35, Theorem 2.2] implies that $f(T)$ obeys the generalized a-Weyl's theorem. □

3. Weyl's Theorem for Algebraically Totally $p-(\alpha, \beta)$-Normal Operators

Lemma 3.1. *Let T be an algebraically totally $p-(\alpha, \beta)$-normal operator. Then T has SVEP.*

Proof. If T is algebraically totally $p-(\alpha, \beta)$-normal operator, then $p(T)$ is totally $p-(\alpha, \beta)$-normal operator for some nonconstant polynomial p. Since $p(T)$ has SVEP. It follows from [22] that T has SVEP. □

Lemma 3.2. *Let T be a quasinilpotent algebraically totally $p-(\alpha, \beta)$-normal operator. Then T is nilpotent.*

Proof. Assume that $p(T)$ is totally $p-(\alpha, \beta)$-normal for some nonconstant polynomial p. Since $\sigma(p(T)) = p(\sigma(T))$, the operator $p(T) - p(0)$ is quasinilpotent. Thus Lemma 2.4 implies that

$$cT^m(T-\lambda_1)(T-\lambda_2)\dots(T-\lambda_n) \equiv p(T) - p(0) = 0,$$

where $m \geq 1$. Since $T - \lambda_i$ is invertible for every $\lambda \neq 0$, we must have $T^m = 0$. □

Theorem 3.3. *Let T be an algebraically totally $p-(\alpha, \beta)$-normal operator. Then T is an isoloid.*

Proof. Let $\lambda \in \sigma(T)$. Using the Riesz idempotent E, we can represent T as a direct sum $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$, where $\sigma(T_1) = \{0\}$ and $\sigma(T_2) = \sigma(T) \setminus \{\lambda\}$. Since T is algebraically totally $p-(\alpha, \beta)$-normal operator, $p(T)$ is totally $p-(\alpha, \beta)$-normal operator for some nonconstant polynomial p. So $\sigma(p(T_1)) = \{p(\lambda)\}$. Therefore $p(T_1) - p(\lambda)$ is quasinilpotent. Since $p(T_1)$ is totally $p-(\alpha, \beta)$-normal operator, from Lemma 2.4 that $p(T_1) - p(\lambda) = 0$. Put $q(z) = p(z) - p(\lambda)$. Then $q(T_1) = 0$, and hence T_1 is algebraically totally $p-(\alpha, \beta)$-normal, it follows from Theorem 3.2 that $T_1 - \lambda$ is nilpotent. Therefore $\lambda \in \pi_0(T_1)$, and hence $\lambda \in \pi_0(T)$. So T is isoloid. □

Theorem 3.4. *Weyl's theorem holds for T when T^* is an algebraically totally $p-(\alpha, \beta)$-normal operator.*

Proof. Suppose that $\lambda \in \sigma(T) \setminus \sigma_w(T)$. Then $T - \lambda$ is Weyl and not invertible. We claim that $\lambda \in \partial\sigma(T)$. Let λ be an interior point of $\sigma(T)$. Then, there exists a neighbourhood U of λ such that $\dim \ker(T - \mu) > 0$ for all $\mu \in U$. It follows from [15] that T does not have SVEP. On the other hand, since $p(T)$ is totally $p-(\alpha, \beta)$-normal for some nonconstant polynomial p, it follows from Lemma 3.1 that T has SVEP, which is a contradiction. Therefore $\lambda \in \partial\sigma(T) \setminus \sigma_w(T)$, and it follows from the punctured neighbourhood theorem that $\lambda \in \pi_{00}(T)$. Conversely, suppose that $\pi_{00}(T)$, using the Riesz idempotent E, we can represent T as the direct sum $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$, where $\sigma(T_1) = \{0\}$ and $\sigma(T_2) = \sigma(T) \setminus \{\lambda\}$.

We consider two cases:

(i) When $\lambda = 0$. Then T_1 is algebraically totally $p-(\alpha, \beta)$-normal and quasinilpotent. It follows from Lemma 3.2 that T_1 is nilpotent. Since $\ker(T_1)$ is infinite dimensional, then $0 \notin \pi_{00}(T)$, which is a contradiction. So $\dim \operatorname{ran}(E) < \infty$. So it follows that T_1 is Weyl. But since T_2 is invertible, we can conclude that T is Weyl. Therefore $0 \in \sigma(T) \setminus \sigma_w(T)$.

(ii) When $\lambda \neq 0$. Then by the proof of Lemma 3.2, $T_1 - \lambda$ is nilpotent. Since $\lambda \in \pi_{00}(T)$, $T_1 - \lambda$ is an operator on the finite dimensional space $\operatorname{ran}(T)$. So $T_1 - \lambda$ is Weyl. Since $T_2 - \lambda$ is invertible, $T - \lambda$ is Weyl. □

Theorem 3.5. *Let T be an algebraically totally $p-(\alpha, \beta)$-normal operator. Then Weyl's theorem holds for $f(T)$ for any function f analytic in the neighbourhood of the spectrum $\sigma(T)$ of T.*

Proof. First, we show that $f(\sigma_w(T)) = \sigma_w(f(T))$ for all functions f analytic in the neighbourhood of $\sigma(T)$ of T. It is enough to prove $f(\sigma_w(T)) \subseteq \sigma_w(f(T))$, since the other inequality is always true. Let $\lambda \notin \sigma_w(f(T))$. Then $f(T) - \lambda$ is Weyl and $f(T) - \lambda = c(T - \alpha_1)(T - \alpha_2)\ldots(T - \alpha_n)g(T)$, where $c, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ and $g(T)$ is invertible. Since the operators on the right side commute, every $T - \alpha_i$ is Fredholm. Since T is algebraically totally $p-(\alpha, \beta)$-normal, T has SVEP by Lemma 3.1. It follows from [1] that $i(T - \alpha_i) \leq 0$ for each $i = 1, 2, \ldots, n \leq$. Therefore $\lambda \notin f(\sigma_w(T))$, and hence $f(\sigma_w(T)) = \sigma_w(f(T))$. From [23], we know that if T is isoloid, then $f(\sigma(T) \setminus \pi_{00}(T)) = \sigma(f(T)) \setminus \pi_{00}(f(T))$, since T is isoloid by Lemma 3.3 and Weyl's theorem holds for T,

$$\begin{aligned} \sigma(f(T)) \setminus \pi_{00}(f(T)) &= f(\sigma(T) \setminus \pi_{00}(T)) \\ &= f(\sigma_w(T)) = \sigma_w(f(T)), \end{aligned}$$

which implies that Weyl's theorem holds for $f(T)$. □

Definition 3.6. For $T \in B(H)$, $\lambda \in \sigma(T)$ is said to be a *regular point* if there exists $S \in B(H)$ such that $T - \lambda = (T - \lambda)S(T - \lambda)$. If every isolated point of $\sigma(T)$ is a *regular point*, then T is called a *reguloid*.

Following lemma is used to prove the corollary below.

Lemma 3.7 ([18]). *$T - \lambda$ has a closed range, if and only if $T - \lambda = (T - \lambda)S(T - \lambda)$.*

Corollary 3.8. *If T is a totally (α, β)-normal operator, then T is a reguloid.*

Proof. Let λ_0 be an isolated point of $\sigma(T)$. Using Reisz idempotent E_λ, we can represent T as a direct sum $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$, where $\sigma(T_1) = \{0\}$ and $\sigma(T_2) = \sigma(T) \setminus \{\lambda_0\}$.

Since T_1 is also (α, β)-normal operator, it follows from Lemma 2.4 that $T_1 = \lambda_0$. Therefore by Theorem 2.5, $H = E(H) \oplus E(H)^{\perp} = \ker(T - \lambda_0) \oplus \ker(T - \lambda_0)^{\perp}$. Hence $T = \lambda_0 \oplus T_2$. Therefore $T - \lambda_0 = 0 \oplus (T_2 - \lambda_0)$ and hence

$$\operatorname{ran}(T - \lambda_0) = (T - \lambda_0)(H) = 0 \oplus (T_2 - \lambda_0)(\ker(T - \lambda_0)^{\perp}).$$

Since $T_2 - \lambda_0$ is invertible, $T - \lambda_0$ has a closed range. □

Theorem 3.9. *Let $T \in B(H)$ be a totally (α, β)-normal operator. Then $f(T)$ satisfies Browder's theorem for each analytic function f in a neighbourhood of $\sigma(T)$ and we have*

$f(\sigma(T) \setminus \Pi_0(T)) = f(\sigma_b(T)) = \sigma_b(f(T)) = \sigma(f(T)) \setminus \Pi_0(f(T)) = f(\sigma(T)) \setminus \Pi_0$ $(f(T))$, *and* $f(\sigma_{Bw}(T)) = \sigma_{Bw}(f(T))$.

Proof. It is known that operators with SVEP satisfy Browder's theorem [14]. Then $f(T)$ satisfies Browder's theorem. Since $f(T)$ satisfies Browder's theorem,

$$
\begin{aligned}
f(\sigma(T) \setminus \Pi_0(T)) &= f(\sigma_b(T)) \\
&= \sigma_b(f(T)) \\
&= \sigma(f(T)) \setminus \Pi_0(f(T)) \\
&= f(\sigma(T)) \setminus \Pi_0(f(T)) \\
&= f(\sigma(T)) \setminus \Pi_0(f(T)),
\end{aligned}
$$

and $f(\sigma_{Bw}(T)) = \sigma_{Bw}(f(T))$.

This completes the proof. □

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